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On long waves on a rotating earth

By Ll. G. CHAMBERS
Department of Applied Mathematics, University College of North Wales, Bangor

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General solutions are obtained for the long gravity wave equation from solutions of the wave equation in three dimensions. The method is applied to the theory of Tsunamis. It is also suggested that these waves are the cause of the shelf oscillations observed at Guadelupe.

1. Introduction

The equations associated with the classical long wave theory have been given by Proudman (1953), but for convenience will be reproduced here.

It is assumed that $\mathbf{q}$, the fluid velocity, is independent of depth, that h, the sea depth, is constant and that the fluid is uniform. It is further assumed that variations in the Coriolis parameter are negligible and the surface of the earth is sensibly plane. Clearly these are idealizations, but nevertheless a comparison can be made between the cases of zero and non-zero Coriolis parameter Ω and an indication obtained of the effect of the introduction of Ω into the more exact and complicated equations.

The horizontal equation of motion is

$$\frac{\partial \mathbf{q}}{\partial t} + \Omega \mathbf{k} \times \mathbf{q} = -g \nabla \zeta, \tag{1}$$

where $\mathbf{q}$ is the horizontal fluid velocity, ζ is the elevation of the free surface above its mean level, $\underline{\underline{\nabla}}$ the two-dimensional gradient operator, $\mathbf{k}$ unit vector vertically upwards and Ω the Coriolis parameter ($\Omega = 2\omega_0 \sin\alpha$ where ω_0 is the angular velocity of the earth and α the north latitude). The continuity equation is

$$\nabla . \mathbf{q} = -\frac{1}{h}\frac{\partial \zeta}{\partial t}. \tag{2}$$

It can be seen that a solution is given by

$$\mathbf{q} = \frac{\partial}{\partial t}\nabla A - \Omega(\mathbf{k} \times \nabla A). \tag{3}$$

$$\zeta = -h\left(\frac{1}{gh}\frac{\partial^2 A}{\partial t^2} - \frac{\Omega^2}{gh}A\right), \tag{4}$$

where

$$\nabla^2 A - \frac{\Omega^2}{gh}A - \frac{1}{gh}\frac{\partial^2 A}{\partial t^2} = 0. \tag{5}$$

If ρ is the density of the fluid, it is fairly easily seen that

$$\rho\left(\nabla . \mathbf{q} + \frac{1}{h}\frac{\partial \zeta}{\partial t}\right) \tag{6}$$

represents the rate of creation of fluid. This can be rewritten in terms of A.

$$\rho\frac{\partial}{\partial t}\left[\nabla^2 A-\frac{\Omega^2}{gh}A-\frac{1}{gh}\frac{\partial^2 A}{\partial t^2}\right]. \tag{7}$$

The velocity normal to a rigid surface must be zero. If the normal to the surface is $\mathbf{n}$ and s is measured along the surface, this boundary condition is

$$\frac{1}{\Omega}\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial n}\right)+\frac{\partial A}{\partial s}=0. \tag{8}$$

General solutions of equation (5) (which may be termed the two-dimensional Proca equation) do not appear to have been given, but certain solutions will be obtained in this paper. It will be found convenient to write $gh = c^2$ and $\Omega^2/(gh) = K^2$.

2. Derivation of long gravity wave solutions from three-dimensional wave equation

Let ψ satisfy the three-dimensional wave equation

$$\nabla^2\psi+\frac{\partial^2\psi}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}=0, \tag{9}$$

and the boundary condition

$$\frac{1}{\Omega}\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial n}\right)+\frac{\partial\psi}{\partial s}=0. \tag{10}$$

Let

$$A_e(\mathbf{r})=\int_{-\infty}^{\infty}\cos Kz\,\psi_e(\mathbf{r},z)\,dz, \tag{11}$$

$$A_o(\mathbf{r})=\int_{-\infty}^{\infty}\sin Kz\,\psi_o(\mathbf{r},z)\,dz, \tag{12}$$

where $\mathbf{r}$ is the horizontal radius vector. ψ_e and ψ_o are solutions of equations (9) subject to equation (10). It is easy to see from an integration by parts that $A_e(\mathbf{r})$ and $A_0(\mathbf{r})$ satisfy equation (5) subject to equation (8) provided that the ψ tend to zero as z tends to infinity. Furthermore A_e is even in K and A_o is odd in K.

As an example of the use of these expressions, consider the following problem. A mass ρV_0 of fluid is liberated uniformly over the depth of the fluid at zero time and along the z axis. A physical discussion of this will follow later.

Then

$$\rho\left(\nabla.\mathbf{q}+\frac{1}{h}\frac{\partial\zeta}{\partial t}\right)=\frac{\rho V_0}{h}\delta(\mathbf{r})\,\delta(t), \tag{13}$$

where $\delta(t)$ is the delta function and $\delta(\mathbf{r})$ is the two-dimensional delta function. It follows that

$$\rho\frac{\partial}{\partial t}\left[\nabla^2 A-K^2A-\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}\right]=\frac{\rho V_0}{h}\delta(\mathbf{r})\,\delta(t), \tag{14}$$

whence

$$\nabla^2 A-K^2A-\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}=\frac{V_0}{h}\delta(\mathbf{r})\,H(t), \tag{15}$$

where $H(t)$ is the Heaviside step function. Clearly the solution of equation (15) can be derived from

$$A = \int_{-\infty}^{\infty} \cos(Kz)\,\psi\,dz, \tag{16}$$

where

$$\nabla^2\psi + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{V_0}{h}\delta(\mathbf{r})\,\delta(z)\,H(t), \tag{17}$$

and the solution is to represent an outgoing wave. The appropriate solution can be seen from Stratton (1941) to be

$$\psi = -\frac{V_0}{4\pi h}\frac{H(t-R/c)}{R}, \tag{18}$$

where

$$R^2 = r^2 + z^2. \tag{19}$$

Applying the transformation (16) to the solution (18)

$$A = -\frac{V_0}{2\pi h}\int_0^{\infty}\frac{\cos(Kz)\,H(t-R/c)}{R}\,dz, \tag{20}$$

using the even property. So

$$A = -\frac{V_0}{2\pi h}\int_0^{(c^2t^2-r^2)^{\frac{1}{2}}}\frac{\cos Kz}{R}\,H\left(t-\frac{r}{c}\right)dz.$$

It will be convenient to write $c^2t^2 - r^2 = b^2$ and $t^* = t - r/c$. With this notation, it follows from equation (4) that

$$\zeta = \frac{V_0}{2\pi gh}\left\{\frac{\partial^2}{\partial t^2}\left[\int_0^b \frac{\cos Kz}{R}\,H(t^*)\,dz\right] + \Omega^2\int_0^b\frac{\cos Kz}{R}\,H(t^*)\,dz\right\}. \tag{21}$$

Now the first term inside the curly bracket on the right of (21) becomes

$$\frac{\partial}{\partial t}\left[\frac{c^2t}{b}\frac{\cos Kb}{ct}H(t^*)\right] + \int_0^b\frac{\cos Kz}{R}\,dz\,\delta(t^*). \tag{22}$$

The second term of (22) vanishes because the coefficient of $\delta(t-r/c)$ vanishes when $t = r/c$, and so (22) becomes

$$\frac{c}{b}\cos Kb\,\delta(t^*) + c^3t\left[-K\frac{\sin Kb}{b^2} - \frac{\cos Kb}{b^3}\right]H(t^*).$$

Clearly in the first term $\cos Kb$ may be replaced by unity, whence it follows that

$$\zeta = \frac{V_0}{2\pi gh}\left\{\left[\Omega^2\int_0^b\frac{\cos Kz}{R}\,dz\,H(t^*) - c^3t\left(\frac{K\sin Kb}{b^2} + \frac{\cos Kb}{b^3}\right)\right]H(t^*) + (c/b)\,\delta(t^*)\right\}. \tag{23}$$

$$\nabla A = -\frac{V_0}{2\pi h}\left[\frac{\mathbf{r}}{b}\frac{\cos Kb}{ct}\right]H(t^*), \tag{24}$$

the term in $\delta(t-r/c)$ vanishing for the same reason as before. On evaluation of $\partial(\nabla A)/\partial t$, it follows that

$$\mathbf{q} = \frac{V_0}{2\pi h}\left[\frac{\mathbf{r}}{b}\frac{1}{ct}\right]\delta(t^*) - \frac{V_0}{2\pi h}\left[\frac{\Omega\mathbf{r}}{b^2}\sin Kb + \frac{\mathbf{r}}{ct^2}\frac{2c^2t^2-r^2}{b^3}\cos Kb + \Omega\frac{\mathbf{k}\times\mathbf{r}}{b}\frac{\cos Kb}{ct}\right]H(t^*). \tag{25}$$

The formula for ζ and $\mathbf{q}$ for a non-rotating system are given from equations (23) and (24) by putting Ω (and K) zero. Considering equations (23) and (24) it will be seen that the initial effect, represented by the term involving $\delta(t-r/c)$ is unchanged. However, the behaviour of the 'tail' is different. If Ω is zero the shape of the tail is given by

$$\zeta = -\frac{V_0}{2\pi g h}\frac{c^3 t}{b^3}, \tag{26}$$

$$\mathbf{q} = -\frac{V_0}{2\pi h}\frac{\mathbf{r}}{ct^2}\frac{2c^2t^2-r^2}{b^3}, \tag{27}$$

and if Ω is non-zero, that is, if rotation is taken into account, the shape of the 'tail' is given by

$$\zeta = \frac{V_0}{2\pi h}\left\{\Omega^2\int_0^b \frac{\cos Kz}{R}dz - c^3t\left(\frac{K\sin Kb}{b^2}+\frac{\cos Kb}{b^3}\right)\right\}, \tag{28}$$

$$\mathbf{q} = -\frac{V_0}{2\pi h}\left[\frac{\mathbf{r}}{ct^2}\frac{2c^2t^2-r^2}{b^3}\cos Kb + \frac{\Omega\mathbf{r}}{b^2}\sin Kb + \Omega\frac{\mathbf{k}\times\mathbf{r}}{b}\frac{\cos Kb}{ct}\right]. \tag{29}$$

As mentioned earlier, a physical interpretation may be obtained for this problem. (I am indebted to a referee for this suggestion.) When earthquakes take place, one possible manifestation is a sudden rising (or lowering) of the sea bed. In effect this is equivalent to the creation (or destruction) of a certain volume of water. Such disturbances give rise to a particularly violent form of long wave, called a Tsunami.

As pointed out previously, the rotation does not affect the initial peak. This is not surprising as the initial delta function merely represents the propagation of the sudden creation. It is in the 'tail' that a difference manifests itself, as time increases. It is stated in Magnus & Oberhettinger (1948) that

$$\int_0^\infty \frac{\cos Kz}{R}dz = \frac{1}{2}\int_{-\infty}^\infty \frac{e^{iKz}}{R}dz = K_0(Kr)$$

and so for the tail

$$\zeta = \frac{V_0}{2\pi g h}\left[\Omega^2 K_0(Kr) - \Omega^2\int_b^\infty \frac{\cos Kz}{R}dz - c^3t\left(\frac{K\sin Kb}{b^2}+\frac{\cos Kb}{b^3}\right)\right]. \tag{30}$$

If Ω is zero, then ζ tends to zero as t tends to infinity. On the other hand, if Ω is non-zero, ζ tends to

$$\frac{K^2V_0}{2\pi}K_0(Kr), \tag{31}$$

which is finite. Now, the 'flat earth' assumption has been made here and so the details of this elevation are obviously not correct. However, the implication is that the elevation decays more slowly in the presence of a rotation than it does otherwise.

If we consider the particle velocity as given by (29), two features arise. First, there is a fluid velocity which is normal to the radius vector to the point of fluid creation represented by the term involving $\mathbf{k}\times\mathbf{r}$ which except in the stages immediately after the arrival of the Tsunami is of the same order as the fluid velocity in the direction of the radius vector. Secondly, it can be seen that for large times, greater than Ω^{-1} say, after the arrival, the rotation causes a less rapid decay. The first term is $O(t^{-3})$, the second and third are $O(t^{-2})$ and they only exist when there is rotation.

3. Shelf oscillations at Guadelupe

In a paper dealing with the diffraction of long waves on a rotating earth in the presence of a semi-infinite barrier Crease (1956) suggested that were there a barrier of finite length—such as an island in mid-ocean, energy might be trapped in the form of a wave progressing round the barrier in a clockwise direction. Many approximations for the shape of an island spring to mind, such as a straight-line segment, or an ellipse but the properties of the solutions of the wave equation associated with such geometries are not really well known and so for the purpose of this paper, the island will be assumed to be of circular shape. It is to be hoped that the solution of problems associated with an island of such a shape will be representative of the solutions associated with islands of more awkward shapes.

Assuming an exponential variation in time of $e^{i\sigma t}$ it can easily be seen that equations (7) and (8) are equivalent to

$$\nabla^2\zeta + k^2\zeta = 0, \tag{32}$$

where

$$k^2 = \frac{\sigma^2-\Omega^2}{gh} > 0, \tag{33}$$

together with the boundary condition

$$i\frac{\partial\zeta}{\partial s} + \gamma\frac{\partial\zeta}{\partial n} = 0, \tag{34}$$

where

$$\gamma = \sigma/\Omega = p^{-1} \quad (0 < p < 1). \tag{35}$$

It is clearly more convenient to use, as is now possible, the physical quantity ζ, the surface elevation rather than the potential A.

For the circle $r = a$, it is convenient to express equation (32) in the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\zeta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\zeta}{\partial\phi^2} + k^2\zeta = 0, \tag{36}$$

and equation (34) in the form

$$\frac{i}{a}\frac{\partial\zeta}{\partial\phi} + \gamma\frac{\partial\zeta}{\partial r} = 0. \tag{37}$$

Consider first of all the free modes; i.e. wave configurations which can exist in the absence of any exciting waves. A solution of equation (36) which has no singularities outside the circle is

$$\zeta_n = Y_n(kr)\, e^{-in\phi}. \tag{38}$$

The behaviour of this for r at infinity, apart from a $r^{-\frac{1}{2}}$ factor, is that of a standing wave. Thus no energy will be lost as no wave is being propagated out and energy will be trapped by the island. Substituting expression (38) in equation (37)

$$(i/a)\{-inY_n(ka)\} + \gamma k Y_n'(ka) = 0,$$

or

$$kaY_n'(ka) + pnY_n(ka) = 0. \tag{39}$$

Thus for a field such as that given by equation (38) to exist, ka must obey equation (39). Thus equation (39) in effect defines the values of σ, the eigen frequencies. It will be observed that if n is positive the elevation defined by (38) rotates with angular velocity (σ/n). If n is positive, this is an anticlockwise rotation; if n is negative it is a clockwise rotation. Thus energy may be trapped either in a clockwise motion or an anticlockwise motion. It will be observed from equation (39) that the eigen frequencies for $n = +N$ are different from those associated with $n = -N$. Furthermore, when $p = 0$ equation (39) reduces to $Y_n'(ka) = 0$, and when $p = 1$ to $Y_{n-1}(ka) = 0$. The roots of (39) will be between these. If the mth positive root of equation (39) is $k_{nm}a$, then the eigen frequencies associated with the system are σ_{nm}, where

$$\sigma_{nm}^2 = \Omega^2 + ghk_{nm}^2. \tag{40}$$

It will be noticed that a solution is possible for $n = 0$, that is, energy can be trapped in a mode which does not rotate with respect to the island. For this case equation (39) becomes $Y_0'(ka) = 0$, which is equivalent to

$$Y_1\left(a\left\{\frac{\sigma_{0m}^2 - \Omega^2}{gh}\right\}^{\frac{1}{2}}\right) = 0. \tag{41}$$

More generally equation (39) can be rewritten as

$$\frac{\sigma_{nm}}{\Omega}\, a\left\{\frac{\sigma_{nm}^2 - \Omega^2}{gh}\right\}^{\frac{1}{2}} Y_n'\left(a\left\{\frac{\sigma_{nm}^2 - \Omega^2}{gh}\right\}^{\frac{1}{2}}\right) + nY_n\left(a\left\{\frac{\sigma_{nm}^2 - \Omega^2}{gh}\right\}^{\frac{1}{2}}\right) = 0,$$

which, putting $\gamma_{nm} = \sigma_{nm}/\Omega$, can be written as

$$\gamma_{nm}(\gamma_{nm}^2 - 1)^{\frac{1}{2}} \mu Y_n'(\mu\{\gamma_{nm}^2 - 1\}^{\frac{1}{2}}) + nY_n(\mu\{\gamma_{nm}^2 - 1\}^{\frac{1}{2}}) = 0, \tag{42}$$

where $\mu = (\Omega a)/\sqrt{(gh)}$, equation (39) becoming

$$Y_1(\mu\{\gamma_{0m}^2 - 1\}^{\frac{1}{2}}) = 0. \tag{43}$$

Shelf oscillations with periods of the order of half or quarter of an hour or so have been observed on Guadalupe, and mentioned by Munk (1958) and Munk, Snodgrass & Tucker (1959). Such oscillations are generally thought to be caused by meteorological factors. Guadalupe is an island off Mexico in the Pacific Ocean with a latitude of 29° N, a greatest dimension of the order of 100 km, and the order of the depth of the sea about it is 1000 m. This corresponds to a characteristic velocity $\sqrt{(gh)}$ of 99·05 m/sec. The Coriolis parameter Ω is

$$2\cdot 2\pi/24 \times 3600 \cos 29° = 0\cdot 127 \times 10^{-3}\,\text{sec}^{-1}.$$

If Guadalupe is idealized into a circle of radius 50 km, the non-dimensional parameter μ which measures the radius of the island is

$$\frac{0\cdot 127 \times 10^{-3}}{99\cdot 05}\, 50 \times 10^3 = 64 \times 10^{-3}.$$

Consider first of all the non-rotating modes, $n = 0$. The first root of $Y_1(x) = 0$ is 2·20. So

$$\gamma_{01}^2 = 1 + \left(\frac{2\cdot 2}{64} \times 10^3\right)^2. \tag{44}$$

Substantially
$$\gamma_{01} = \frac{2 \cdot 2 \times 10^3}{64} = 34 \cdot 4,$$

and so
$$\sigma_{01} = 34 \cdot 4 \times 0 \cdot 127 \times 10^{-3} \sec^{-1},$$
$$= 4 \cdot 35 \times 10^{-3} \sec^{-1}, \tag{45}$$

which corresponds to a period of about 22 min. For the higher roots γ_{0n} the periods will be correspondingly less. As only the order of magnitude is of interest, an approximate solution of equation (41) in the general case will be sufficient. It can easily be seen by dividing by γ_{nm} that if n is not large an approximate solution is given by the solution of

$$Y_n' \left(\frac{\sigma_{nm} a}{\sqrt{(gh)}} \right) = 0. \tag{46}$$

This is because μ is small compared to unity and $(\gamma_{nm}^2 - 1)^{\frac{1}{2}} \mu$ is of the order of unity and (Ω/σ) is very small compared to unity in the range of interest. Thus if $n = \pm m$ where m is not large, the eigenfrequencies are very nearly the same.

In particular for $n = \pm 1$, the first zero of $Y_1'(x)$ is $3 \cdot 68$. This corresponds to

$$\sigma_{-1,1} = \sigma_{11} = \frac{3 \cdot 68 \sqrt{(gh)}}{a},$$
$$= 7 \cdot 44 \times 10^{-3} \sec^{-1}, \tag{47}$$

corresponding to a period of about 14 min. For the higher roots the periods will be correspondingly reduced.

Thus, the periods for the shelf oscillations obtained on Guadelupe are of the same order as those predicted by the idealized theory discussed in this paper (because of the idealizations involved any actual numerical agreement will be fortuitous) and it may well be that they are caused by a mechanism of this nature. If there is any disturbance of the relevant frequencies, caused meteorologically or otherwise, this will tend to be trapped by the island, and may involve a trapping of energy in waves progressing round the island in either direction, or, as indicated above, without any angular progression at all. This is borne out in the results of Proudman (1914) concerning a plane wave incident on a circular island. If the frequency associated with the plane wave or indeed with any incident field, is one of the eigenfrequencies associated with the circular island, resonance takes place and infinities occur in the reflected wave.

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